ON THE PERMANENT AXES OF MOTION FOR A HEAVY GYROSTAT NEAR AN IMMOVABLE POINT

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PMM Vol.25, No.5, 1961, pp. 941-945

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(Received April 28, 1961)

A gyrostat [1] is a mechanical system S which consists of an invariable part S_1 and other bodies S_2, variable or rigid, but connected not invariably with S_1. At the same time it is necessary that the motion of bodies S_2 relative to S_1 does not change the mass geometry of system S. Permanent motions of a balanced gyrostat under constant gyrostatic moment were investigated in detail by Volterra [2]. Zhukovskii presented [3] a geometric interpretation of this motion.

Below are determined permanent motions of a heavy gyrostat near an immovable point by a scheme analogous to that used by Mlodzeevskii [4] for determination of permanent motions of a rigid body near an immovable point.

Let the invariable part S_1 of the gyrostat be fixed at one of its points O which we will take as the origin of two coordinate systems: a stationary system $O\xi\eta\zeta$ with the vertically directed axis $O\zeta$, and a moving system $Oxyz$, the axes of which are directed along the principal axes of inertia of the gyrostat about the point O. On the strength of a theorem for addition of velocities, the momentum of the gyrostat about the point O can be resolved into two components: the vector **K**, referred to the whole system S which results from its translational motion, and the vector **k** governed by the motion of S_2 relative to S_1. We will assume that the gyrostatic momentum **k** is constant in magnitude and direction relative to S_1. The projections of this vector on the moving axes will be denoted by k_x, k_y, k_z, while the projections of **K** on the same axes as

$$K_x = Ap, \quad K_y = Bq, \quad K_z = Cr$$

Here p, q, r denote the projections of the vector ω, the instantaneous angular velocity of the gyrostat, on the stationary axes, while A, B, C

are the principal moments of inertia of the gyrostat about point O.

By the momentum theorem we obtain the following equations of motion for a heavy gyrostat near an immovable point:

$$\begin{aligned}
&A\frac{dp}{dt}+(C-B)\,qr+qk_z-rk_y=P\,(z_0\gamma_2-y_0\gamma_3)\\
&B\frac{dq}{dt}+(A-C)\,rp+rk_x-pk_z=P\,(x_0\gamma_3-z_0\gamma_1)\\
&C\frac{dr}{dt}+(B-A)\,pq+pk_y-qk_x=P\,(y_0\gamma_1-x_0\gamma_2)
\end{aligned}\tag{1}$$

Here x_0, y_0, z_0 denote the coordinates of the center of gravity of the gyrostat in the moving system of coordinates, P is the weight of the gyrostat, and γ_1, γ_2, γ_3 are direction cosines of the axis $O\zeta$ relative to the moving coordinates satisfying the kinematic equations of Poisson

$$\frac{d\gamma_1}{dt}=r\gamma_2-q\gamma_3,\qquad \frac{d\gamma_2}{dt}=p\gamma_3-r\gamma_1,\qquad \frac{d\gamma_3}{dt}=q\gamma_1-p\gamma_2\tag{2}$$

Let us consider under what conditions the gyrostat will rotate permanently, i.e. rotate about axes which are fixed in the gyrostat. Let such rotations exist. Then

$$p=a\omega,\qquad q=b\omega,\qquad r=c\omega$$

where a, b, c, are the constant direction cosines of the required axis in the xyz-system ofcoordinates. In this case the equations of motion for the gyrostat become

$$\begin{aligned}
&Aa\frac{d\omega}{dt}+(C-B)\,bc\omega^2+(bk_z-ck_y)\,\omega=P\,(z_0\gamma_2-y_0\gamma_3)\\
&Bb\frac{d\omega}{dt}+(A-C)\,ac\omega^2+(ck_x-ak_z)\,\omega=P\,(x_0\gamma_3-z_0\gamma_1)\\
&Cc\frac{d\omega}{dt}+(B-A)\,ab\omega^2+(ak_y-bk_x)\,\omega=P\,(y_0\gamma_1-x_0\gamma_2)
\end{aligned}\tag{3}$$

$$\frac{d\gamma_1}{dt}=\omega\,(c\gamma_2-b\gamma_3),\qquad \frac{d\gamma_2}{dt}=\omega\,(a\gamma_3-c\gamma_1),\qquad \frac{d\gamma_3}{dt}=\omega\,(b\gamma_1-a\gamma_2)\tag{4}$$

Equations (4) possess two integrals

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1\tag{5}$$

$$a\gamma_1+b\gamma_2+c\gamma_3=\nu\tag{6}$$

Multiplying Equations (3) by γ_1, γ_2, γ_3; x_0, y_0, z_0 and k_x, k_y, k_z, respectively, and adding, we obtain

$$(Aa\gamma_1 + Bb\gamma_2 + Cc\gamma_3)\frac{d\omega}{dt} + [(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3]\,\omega^2 + \\ + [(bk_z - ck_y)\,\gamma_1 + (ck_x - ak_z)\,\gamma_2 + (ak_y - bk_x)\,\gamma_3]\,\omega = 0 \tag{7}$$

$$(Aax_0 + Bby_0 + Ccz_0)\frac{d\omega}{dt} + [(C-B)\,bcx_0 + (A-C)\,acy_0 + (B-A)\,abz_0]\,\omega^2 + \\ + [(bk_z - ck_y)\,x_0 + (ck_x - ak_z)\,y_0 + (ak_y - bk_x)\,z_0]\,\omega = 0 \tag{8}$$

$$(Aak_x + Bbk_y + Cck_z)\frac{d\omega}{dt} + [(C-B)\,bck_x + (A-C)\,ack_y + (B-A)\,abk_z]\,\omega^2 - \\ - P\,[(z_0\gamma_2 - y_0\gamma_3)\,k_x + (x_0\gamma_3 - z_0\gamma_1)\,k_y + (y_0\gamma_1 - x_0\gamma_2)\,k_z] = 0 \tag{9}$$

There are two similar equations (7) and (8) for ω. The coefficients of these equations must be proportional. If the constant coefficients of Equations (8) are denoted by L, M, N, we obtain

$$\frac{Aa\gamma_1 + Bb\gamma_2 + Cc\gamma_3}{L} = \frac{(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3}{M} \\ = \frac{(bk_z - ck_y)\,\gamma_1 + (ck_x - ak_z)\,\gamma_2 + (ak_y - bk_x)\,\gamma_3}{N}$$

From these one can obtain two independent equations

$$M\,[Aa\gamma_1 + Bb\gamma_2 + Cc\gamma_3] = L\,[(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3] \tag{10}$$

$$N\,[(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3] \\ = M\,[(bk_z - ck_y)\,\gamma_1 + (ck_x - ak_z)\gamma_2 + (ak_y - bk_x)\,\gamma_3] \tag{11}$$

Differentiating (10) with respect to t, taking into account Equations (4) and the integral (6), we obtain

$$-M\,[(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3] \\ = L\,(Aa\gamma_1 + Bb\gamma_2 + Cc\gamma_3) - L\nu\,(Aa^2 + Bb^2 + Cc^2) \tag{12}$$

Solving the system of equations (10), (11), (12), we find

$$Aa\gamma_1 + Bb\gamma_2 + Cc\gamma_3 = \frac{L^2\nu\,(Aa^2 + Bb^2 + Cc^2)}{L^2 + M^2}$$

$$(C-B)\,bc\gamma_1 + (A-C)\,ac\gamma_2 + (B-A)\,ab\gamma_3 = \frac{LM\nu\,(Aa^2 + Bb^2 + Cc^2)}{L^2 + M^2}$$

$$(bk_z - ck_y)\,\gamma_1 + (ck_x - ak_z)\,\gamma_2 + (ak_y - bk_x)\,\gamma_3 = \frac{LN\nu\,(Aa^2 + Bb^2 + Cc^2)}{L^2 + M^2} \tag{13}$$

Thus, for γ_1, γ_2, γ_3 there are five algebraic equations (5), (6), (13) with constant coefficients, four of which are linear. The linear equations (6), (13) can be considered as a system of homogeneous equations relative to γ_1, γ_2, γ_3, ν. The determinant of this system must be equal to zero; constructing it we obtain

$$D = (Aa^2 + Bb^2 + Cc^2)\,[(C-B)\,bck_x + (A-C)\,ack_y + (B-A)\,abk_z] \times \times \left[1 - \frac{L}{L^2+M^2}(Aa^2+Bb^2+Cc^2)(ax_0+by_0+cz_0)\right] = 0 \qquad (14)$$

One condition for the existence of a nontrivial solution will be

$$(C-B)\,k_x bc + (A-C)\,k_y ac + (B-A)\,k_z ab = 0 \qquad (15)$$

We observe that the other condition

$$1 - \frac{L}{L^2+M^2}(Aa^2+Bb^2+Cc^2)(ax_0+by_0+cz_0) = 0$$

yields nothing new compared to (15), and will therefore not be considered.

In satisfying the condition (15) there will be nonzero determinants of third order; for example, such will be the determinant located in the upper left corner of the determinant D. Consequently, γ_1, γ_2, γ_3 can be determined from the system of nonhomogeneous linear equations with constant coefficients, i.e. the problem is to find constant γ_1, γ_2, γ_3. In this case Equations (4) yield

$$\frac{a}{\gamma_1} = \frac{b}{\gamma_2} = \frac{c}{\gamma_3} \qquad (16)$$

This indicates that the investigated permanent axis will be vertical.

On the strength of (16) the coefficients of ω and ω^2 in Equation (7) are equal to zero, and since, generally speaking, the angular velocity ω is finite then

$$d\omega\,/\,dt = 0 \qquad (17)$$

i.e. the gyrostat rotates uniformly about the vertical permanent axis.

Taking this into consideration as well as condition (15), Equation (9) yields

$$(y_0k_z - z_0k_y)\,a + (z_0k_x - x_0k_z)\,b + (x_0k_y - y_0k_x)\,c = 0 \qquad (18)$$

But then Equation (8) gives an additional condition for direction cosines

$$(C-B)\,x_0bc + (A-C)\,y_0ac + (B-A)\,z_0ab = 0 \qquad (19)$$

Equation (19) is the equation for the Mlodzeevskii-Staude cone of permanent axes for a rigid body. Equation (15) becomes the cone equation for permanent axes of a balanced gyrostat. Equation (18) is the equation for surface of the vectors $\mathbf{k}(k_x,\ k_y,\ k_z)$ and $\mathbf{r}_0(x_0,\ y_0,\ z_0)$.

The vertical permanent axis must lie simultaneously on these three surfaces. In the general case the surfaces (15), (18), (19) possess a unique common straight line. Its equation in the moving system of coordinates is

$$\frac{x}{(C-B)R_y R_z} = \frac{y}{(A-C)R_z R_x} = \frac{z}{(B-A)R_x R_y} \tag{20}$$

Here R_x, R_y, R_z are the projections of the vector $\mathbf{R} = \mathbf{k} \times \mathbf{r}_0$ on the moving axes.

The angular velocity of permanent motion of the gyrostat can be found from any one of the equations of motion which can be put in the following form:

$$(A-C)(B-A)(C-B)R_x R_y R_z \omega^2 - n(AR_x k_x + BR_y k_y + CR_z k_z)\omega + \\ + nP(AR_x x_0 + BR_y y_0 + CR_z z_0) = 0 \tag{21}$$

Here

$$n = \sqrt{(C-B)^2 R_y^2 R_z^2 + (A-C)^2 R_z^2 R_x^2 + (B-A)^2 R_x^2 R_y^2}$$

It is clear from this that of any two segments of a straight line in (20) only one can be the axis of permanent rotation in the general case, namely that one for which the discriminant of Equation (21) is always positive. (Interchange of the straight line segments results in the interchange of the vectors $\mathbf{k}$ and $\mathbf{r}_0$ which in turn results in a sign change of the free term.) The gyrostat can rotate near such a segment in one or another direction but the angular velocities of these rotations will be different. We will explain these facts assuming that a vertical axis of rotation is given and will determine the conditions for which a permanent rotation about this axis is possible.

The permanent rotation axis of a heavy gyrostat is located in the vertical plane (18) passing through the support point and possessing an invariable location in the gyrostat. In this plane is also located the constant vector $\mathbf{K}$ relative to the gyrostat. In such a case the weight moment $\mathbf{M}_3$ can be counter-balanced by a gyroscopic moment $\mathbf{M} = -(\omega \times (\mathbf{K} + \mathbf{k}))$ since both moments are directed along the straight line perpendicular to the plane (18) which we will call, as in the case of a rigid body (5), the central vertical plane.

The vertical axis of permanent rotation divides the central vertical plane in two half-planes. Then one can state that in order that the equality $\mathbf{M}_3 = -\mathbf{M}$ take place, the center of gravity of the gyrostat must lie in the right central half-plane when looking from the end of the vector $\mathbf{M}_1 = -(\omega \times \mathbf{K})$. Since the center of gravity of the gyrostat is prescribed, this means the following.

Of the two straight line segments forming the axis of permanent rotation, the one relative to which the gyrostat center of gravity is in the right-hand half-plane when viewed from the end of the $\mathbf{M}_1$ vector, must be directed upward.

The reversal of rotation about a given segment only reverses the sign of the gyroscopic moment $\mathbf{M}_2 = -(\omega \times \mathbf{k})$. Previous rotation is possible but the angular velocity will now be different.

Let us consider a number of particular cases.

(a) Let the inertia ellipsoid of the gyrostat be an ellipsoid of rotation, for example $A = B$. Thereby the straight line (20) is located on the principal surface of inertia associated with that principal axis of inertia for which the moment of inertia is not equal to the two remaining ones. In this case the equation for the straight line is given by

$$R_x x + R_y\, y = 0, \qquad z = 0 \tag{22}$$

The angular velocity of the gyrostat for this axis is

$$\omega = P\,\frac{z_0}{k_z} \tag{23}$$

In the case of a spherical ellipsoid of inertia Equations (15) and (19) become identities and the plane (18) for the permanent axes of the gyrostat is obtained. The angular velocity will be given, for example, by the equation

$$\omega = P\,\frac{bz_0 - cy_0}{bk_z - ck_y} \tag{24}$$

Any straight segment of the surface (18) can serve as a permanent axis of rotation, but the rotation can take place in one direction only. The angular velocity of rotation for the axis coinciding with the vector $\mathbf{k}$ is $\omega = \infty$. For the axis passing through the center of gravity of the gyrostat $\omega = 0$, which corresponds to the equilibrium of the gyrostat when its center of gravity occupies the highest or the lowest position.

(b) Let the projections of $\mathbf{k}$ and $\mathbf{r}_0$ on two axes be proportional to each other, for example

$$\frac{x_0}{k_x} = \frac{y_0}{k_y}$$

In this case the cones (15) and (19) are intersected by these axes and touch along the third principal axis which serves as the permanent axis of rotation. The angular velocity of the gyrostat is

$$\omega = P\frac{x_0}{k_x} = P\frac{y_0}{k_y}$$

If the vector **k** of the gyrostatic moment passes through the center of gravity:

$$\frac{x_0}{k_x} = \frac{y_0}{k_y} = \frac{z_0}{k_z}$$

then the cones (15) and (19) coincide, while the plane (18) ceases to exist. Thus, in this case the geometric location of the permanent axes for a heavy gyrostat is the Mlodzeevski-Staude cone (19).

The angular velocity will be determined from one of the following equations:

$$\begin{aligned}(C - B)\, x_0\, bc\, \omega^2 + k_x\, (bz_0 - cy_0)\, \omega &= P\, (bz_0 - cy_0)\, x_0\\ (A - C)\, y_0\, ac\, \omega^2 + k_y\, (cx_0 - az_0)\, \omega &= P\, (cx_0 - az_0)\, y_0\\ (B - A)\, z_0\, ab\, \omega^2 + k_z\, (ay_0 - bx_0)\, \omega &= P\, (ay_0 - bx_0)\, z_0\end{aligned} \tag{25}$$

Unlike for the rigid body, the angular velocity of a heavy gyrostat for the principal axes of inertia is a finite quantity and equals

$$\omega = P\frac{x_0}{k_x} = P\frac{y_0}{k_y} = P\frac{z_0}{k_z}$$

Following Staude [6], we will refer to the half-generators of cone (19), for which Equations (25) give real values of ω, as the admissible conditions of the problem and the remaining ones as not admissible. For an arbitrary value of the gyrostatic moment, the admissible conditions for a heavy gyrostat will be the same as those for a rigid body. However, if the gyrostatic moment is sufficiently large, then the permanent axis of rotation can be any half-generator of the cone (19).

It may happen that all third-order determinants for the system of linear equations (6), (13) are zero while among the determinants of second order there are nonzero ones. This is possible for different particular admissions considered above. Using Equation (5), one may again find a constant solution for γ_1, γ_2, γ_3, i.e. the permanent axis will, as before, be vertical.

A final possibility remains, namely, that there is only one independent equation among the linear equations (6), (13). The permanent axis will not be vertical.

Let the equations of system (13) follow from Equation (6); then the coefficients in these equations must be proportional:

$$\frac{Aa}{a}=\frac{Bb}{b}=\frac{Cc}{c}=\frac{L^2\,(Aa^2+Bb^2+Cc^2)}{L^2+M^2} \tag{26}$$

$$\frac{(C-B)\,bc}{a}=\frac{(A-C)\,ac}{b}=\frac{(B-A)\,ab}{c}=\frac{LM\,(Aa^2+bB^2+Cc^2)}{L^2+M^2} \tag{27}$$

$$\frac{bk_z-ck_y}{a}=\frac{ck_x-ak_z}{b}=\frac{ak_y-bk_x}{c}=\frac{LN\,(Aa^2+bB^2+Cc^2)}{L^2+M^2} \tag{28}$$

Equations (26), (27) can be satisfied either by letting two of the cosines a, b, c equal zero, or by equating two moments of inertia A, B, C and making the cosine at the third moment zero, or by equating all three moments of inertia A, B, C.

But if two moments of inertia are equal to each other, for example $A = B$, then the moving axes x and y can always be chosen in such a way that a or b is zero. One may proceed analogously when $A = B = C$, i.e. all three cases can be reduced to one: one cosine equals unity and the other two are zero, for example

$$a = 1, \qquad b = 0, \qquad c = 0 \tag{29}$$

If one requires, in addition, that the vector **k** of the gyrostatic moment be directed along that of the principal axes of inertia for which the cosine is unity, e.g.

$$k_y = k_z = 0, \qquad k_x = k \tag{30}$$

then Equation (28) will also be satisfied.

On the strength of (29) and (30) we have $L = Ax_0$ $M = N = 0$. Since now $d\omega/dt \neq 0$, then it follows from (8) that $L = 0$, i.e. $x_0 = 0$. Further, Equation (7) becomes

$$A\gamma_1\frac{d\omega}{dt}=0,\ \text{i.e.}\ \gamma_1=0$$

Then the angular velocity of the gyrostat is given by

$$A\frac{d\omega}{dt}=P\,(z_0\gamma_2-y_0\gamma_3) \tag{31}$$

Consequently, since $b = c = 0$, in the given case the gyrostat rotates near one of the principal axes of inertia (in the case considered near the axis Ox), located horizontally; whereas the center of gravity of the gyrostat must be located in the principal inertia plane associated with that principal inertia axis near which the rotation takes place, while the gyrostatic moment must be directed along the axis of rotation. Thus, this is the case of a common physical pendulum.

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Translated by V.C.